11. RUBAN A.I., On laminar separation from the break point of a rigid surface. Uch. Zap. TsAGI, 5, 2, 1974.
12. RUBAN A.I., A numerical method of solving the problem of free interaction. Uch. Zap. TsAGI, 7, 2, 1976.
13. DANIELS P.G., Viscous mixing at a trailing edge. Quart. J. Mech. Appl. Math., 30, 3, 1977.

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ANOMALOUS HYDRODYNAMIC FLUCTUATIONS DURING THE DEVELOPMENT OF THERMAL CONVECTION*

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The spectral functions of the fluctuations of the hydrodynamic variables in an inhomogeneously heated liquid in the region of Rayleigh numbers close to the threshold of convective stability have been calculated using the equations of the correlation theory of thermal fluctuations in non-equilibrium statistical systems. It is shown that anomalous fluctuations, which form the structure of the flow in the regions of supercritical values of the Rayleigh number are of an essentially non-equilibrium nature and are completely accounted for by the long wavelength part of the correlation functions. The results of the calculations are used to analyse the effect of large-scale fluctuations on the Rayleigh scattering of radiation. It is is shown that, in a region where thermal convection develops, they are responsible for a phenomenon which is analogous to the critical opalescence of light during equilibrium phase transitions of the second kind.

The investigation of the dependence of the spectral functions of thermal hydrodynamic fluctuations on the degree of non-equilibrium in statistical systems is of great significance in the development of optical methods for the noise diagnostics of inhomogeneous flows of liquids and gases. Systems which are far removed from thermodynamic equilibrium and, in particular, the fluctuation mechanisms of processes involving the self-organization of flow structures when there is loss of stability are of special interest. A large number of papers (/1-4/, for example) have been concerned with the study of the anomalous hydrodynamic fluctuations which develop close to the thermal convection threshold in a liquid which is heated from below. However, the results obtained in the majority of these papers are contradictory as for example, in /1/ and /2/. This is explained by the previously discussed /5-7/ incompleteness of the theories of non-equilibrium hydrodynamic fluctuation theories which were employed. In this paper an analysis of the anomalous fluctuations during the development of thermal convection is carried out using the solution of the equations of the theory in /6/ which enables one to evaluate the results which have previously been obtained from common positions.

1. Initial equations and formulation of the problem.

Let us consider a one-component inhomogeneous continuous medium which is described by a system of Navier-Stokes-Fourier equations for the mean values of the density n, the hydrodynamic velocity $\mathbf{u}$ and the thermal energy density $e = {}^3/_2\, k_B T$, where k_B is Boltzmann's constant and T is the mean value of the temperature. We shall write this system of equations in the symbolic form:

$$\frac{\partial}{\partial t}\Phi_\nu + \Lambda_\nu[\Phi; \mathbf{r}] = 0, \quad \nu = 0, 1, 2, 3, 4 \tag{1.1}$$

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where $\mathbf{\Phi}=(n, u_1, u_2, u_3, e)$. Let us denote the fluctuations of the hydrodynamic fields by $\delta\Phi_\nu(\nu=0,1,2,3,4)$ and $\delta\mathbf{\Phi}=(\delta n, \delta u_1, \delta u_2, \delta u_3, \delta e)$.

In the linear theory of non-equilibrium thermal fluctuations /6/, the pair correlation functions of the hydrodynamic variables are broken down into two parts

$$\langle\delta\Phi_\mu(t+\tau, \mathbf{r}_1)\,\delta\Phi_\nu(t, \mathbf{r}_2)\rangle=\alpha_{\mu\nu}(t+\tau, \mathbf{r}_1; t, \mathbf{r}_2)+\beta_{\mu\nu}(t+\tau, \mathbf{r}_1; t, \mathbf{r}_2) \tag{1.2}$$

such that $\alpha_{\mu\nu}(t, \mathbf{r}_1; t, \mathbf{r}_2)\sim\delta(\mathbf{r}_1-\mathbf{r}_2)$ and $\beta_{\mu\nu}(t, \mathbf{r}_1; t, \mathbf{r}_2)$ describes the long wavelength spatial statistical links of a non-equilibrium nature. The dynamics of the two-time correlators of the non-equilibrium hydrodynamic fields are determined by the solution of the linearized equations /6/

$$\frac{\partial}{\partial\tau}\begin{Vmatrix}\alpha_{\mu\nu}(t+\tau, t)\\ \beta_{\mu\nu}(t+\tau, t)\end{Vmatrix}+\sum_{\gamma=0}^{4}\Lambda'_{\mu,\nu}[\mathbf{\Phi}; \mathbf{r}_1]\begin{Vmatrix}\alpha_{\gamma\nu}(t+\tau, t)\\ \beta_{\gamma\nu}(t+\tau, t)\end{Vmatrix}=0 \tag{1.3}$$

$$(\Lambda'_{\mu,\gamma}[\mathbf{\Phi}; \mathbf{r}]\,\varphi(\mathbf{r})=\int d\mathbf{r}'\varphi(\mathbf{r}')\,\delta\Lambda_\mu[\mathbf{\Phi}; \mathbf{r}]/\delta\Phi_\gamma(\mathbf{r}'))$$

with the initial conditions

$$\alpha_{\mu\nu}(t, \mathbf{r}_1; t, \mathbf{r}_2)=a_{\mu\nu}(t, \mathbf{r}_1, \mathbf{r}_2), \quad \beta_{\mu\nu}(t, \mathbf{r}_1; t, \mathbf{r}_2)=b_{\mu\nu}(t, \mathbf{r}_1, \mathbf{r}_2) \tag{1.4}$$

Here, $\Lambda_{\mu,\gamma}'$ are linearized Navier-Stokes-Fourier operators, $\delta/\delta\Phi_\gamma(\mathbf{r})$ is a functional derivative and $a_{\mu\nu}$ and $b_{\mu\nu}$ are the single-time correlation functions of the small-scale and large-scale fluctuations.

It follows from kinetic theory /6/ that

$$\begin{aligned}&a_{\mu\nu}(t, \mathbf{r}_1, \mathbf{r}_2)=\delta(\mathbf{r}_1-\mathbf{r}_2)\int d\mathbf{v}\,\Psi_\mu(\mathbf{v}, \mathbf{r}_1)\,\Psi_\nu(\mathbf{v}, \mathbf{r}_2)\,F(t, \mathbf{v}, \mathbf{r}_1)\\&\Psi_0=1;\quad \Psi_k=(v_k-u_k)\,n^{-1}, \quad k=1,2,3;\\&\Psi_4=[m(\mathbf{v}-\mathbf{u})^2\,2^{-1}-e]\,n^{-1}\end{aligned} \tag{1.5}$$

(F is a single-particle distribution function which satisfies the Boltzmann equation). The explicit form of the solution of the Boltmann equation in an approximation which is linear with respect to the gradients of the hydrodynamic variables enables one to obtain

$$\begin{aligned}&a_{00}=\delta(\mathbf{r}_1-\mathbf{r}_2)\,n, \quad a_{kl}=\delta(\mathbf{r}_1-\mathbf{r}_2)p\,(\rho n)^{-1}[\delta_{kl}+P_{kl}p^{-1}]\\&a_{k4}=\delta(\mathbf{r}_1-\mathbf{r}_2)\,n^{-2}q_k, \quad a_{44}=\delta(\mathbf{r}_1-\mathbf{r}_2)\cdot{}^3/_2 n^{-1}(k_BT)^2\\&a_{0k}=a_{04}=0, \; a_{\mu\nu}=a_{\nu\mu}, \; k, \; l=1, \; 2, \; 3\\&P_{kl}=-\eta\left(\frac{\partial u_k}{\partial r_l}+\frac{\partial u_l}{\partial r_k}-\frac{2}{3}\delta_{kl}\frac{\partial u_i}{\partial r_i}\right), \quad q_k=-\lambda\frac{\partial T}{\partial r_k}, \quad p=nk_BT\end{aligned} \tag{1.6}$$

from (1.5). Here, p is the pressure, δ_{kl} is the Kronecker delta, P_{kl} are the components of the stress tensor, q_k are the components of the thermal flux and η and λ are the viscosity and thermal conductivity respectively.

It is impossible to calculate the single-time correlators $b_{\mu\nu}$ of the large scale fluctuations directly. They are defined by the solution of the system of equations /6, 7/

$$\frac{\partial}{\partial t}b_{\mu\nu}+\sum_{\gamma=0}^{4}\{\Lambda'_{\mu,\gamma}[\mathbf{\Phi}; \mathbf{r}_1]\,b_{\gamma\nu}+\Lambda'_{\nu,\gamma}[\mathbf{\Phi}; \mathbf{r}_2]\,b_{\mu\gamma}\}=H_{\mu\nu}[\mathbf{\Phi}; \mathbf{r}_1, \mathbf{r}_2] \tag{1.7}$$

the inhomogeneous terms of which describe the processes involved in the generation of large-scale fluctuations in non-equilibrium systems. The dependences of $H_{\mu\nu}$ on the mean values of the hydrodynamic variables and thermodynamic forces have been obtained in /6/ and /7/ and, in the approximation which is linear with respect to thermodynamic forces, have the form

$$\begin{aligned}H_{\mu\nu}[\mathbf{\Phi}; \mathbf{r}_1, \mathbf{r}_2]=\delta(\mathbf{r}_1-\mathbf{r}_2)\frac{p}{\rho n\eta}\sum_{k,l=1}^{3}\delta_{\mu k}\delta_{\nu l}P_{kl}+\\ \delta(\mathbf{r}_1-\mathbf{r}_2)\cdot\frac{5}{2}k_B\frac{p}{\rho n\lambda}\sum_{k=1}^{3}[\delta_{\mu 4}\delta_{\nu k}+\delta_{\mu k}\delta_{\nu 4}]\,q_k\end{aligned} \tag{1.8}$$

Eqs.(1.2)-(1.8) constitute the basis of the linear correlation theory of non-equilibrium thermal hydrodynamic fluctuations which assumes that the fluctuations are small compared with the mean values of the physical parameters and, consequently, is not applicable in a small neighbourhood of the stability threshold where the susceptibility of the system is anomalously large.

Nevertheless, these equations can be used in a study of the general tendency of the change in the spectral properties of the fluctuations when the state of the system approximates to the stability threshold. We shall make use of them for a statistical description of an inhomogeneously heated liquid which is characterized by a value of the Rayleigh number which is close to the critical value.

Let us consider hydrodynamic fluctuations during the development of thermal convection in an immobile liquid occupying a closed space which is heated from below. Let the size of this closed space in the vertical direction be so small that the effects of compressibility may be neglected and the Oberbeck-Boussinesq simplifications /8/ which are traditional in problems of convective stability can be used. We shall confine ourselves to the case of a stationary temperature distribution in the cavity. In this case the solutions of Eqs.(1.1) of the form

$$\mathbf{u}=0,\quad \nabla T=-a\boldsymbol{\varepsilon},\quad a=\mathrm{const}>0,\quad \boldsymbol{\varepsilon}=\{\varepsilon_x=0,\varepsilon_y=0,\varepsilon_z=1\} \tag{1.9}$$

are the conditions for the mechanical equilibrium of the medium (the Oz-axis is directed opposite to the $\mathbf{g}$ vector for the free-fall acceleration).

Let us change to dimensionless quantities in Eqs.(1.3) and (1.7) by choosing, as the units of measurement of time, distance, velocity, temperature and pressure, the quantities $h^2\nu^{-1}$, h, χh^{-1}, ah, $\rho\nu\chi h^{-2}$ respectively, where $\nu=\eta\rho^{-1}$, $\chi=\lambda(\pi c_p)^{-1}$ are the kinematic viscosity and thermal conductivity and c_p is the heat capacity. In the approximation being considered, the dimensionless quantities Λ_μ have the form

$$\begin{aligned}&\Lambda_0=0,\quad \Lambda_k=\mathrm{Pr}^{-1}\mathbf{u}\cdot\nabla u_k+\nabla^2u_k+\mathrm{Ra}\,\varepsilon_kT,\quad k=1,2,3\\&\Lambda_4=-\mathrm{Pr}^{-1}\mathbf{u}\cdot\nabla T+\mathrm{Pr}^{-1}\nabla^2T,\quad \mathrm{Pr}=\nu\chi^{-1},\quad \mathrm{Ra}=g\beta ah^4(\nu\chi)^{-1}\end{aligned} \tag{1.10}$$

In the case of (1.9), it is only those components of the tensor $H_{\mu\nu}$ in (1.8) which allow for the generation of spatial correlation of the fluctuations of the vertical component of the velocity and temperature

$$H_{34}=H_{43}=\delta(\mathbf{r}_1-\mathbf{r}_2)\cdot{}^5/_2\ QT,\quad Q=k_Ba(\rho\nu\chi)^{-1} \tag{1.11}$$

which differ from zero.

The dimensionless unit Q characterizes the level of intensity of molecular noise in the non-equilibrium stability of the system and has the meaning of the ratio of the thermodynamic force to the force of dissipation.

The formulation of the problem is completed by setting out the boundary conditions for Eqs.(1.3) and (1.7). Subsequently, we shall be considering the case of a closed cavity in the block of a highly conducting solid for which

$$\delta\Phi_\mu(t,\mathbf{r})|_{\mathbf{r}\in s}=0 \tag{1.12}$$

where s is the surface of the cavity.

2. Anomalous fluctuations close to the threshold of convective stability.

In solving the system of Eqs.(1.3), (1.7) taking account of (1.9)-(1.11) subject to conditions (1.4) and (1.12), we shall make use of the fact that the boundary value problem

$$\sum_{\nu=0}^{4}\Lambda'_{\mu,\nu}[\boldsymbol{\Phi};\mathbf{r}]\,Y_\nu(\mathbf{r})=-\gamma Y_\mu(\mathbf{r}),\quad Y_\mu(\mathbf{r})|_{\mathbf{r}\in s}=0 \tag{2.1}$$

for the linearized operators (1.10) is selfadjoint /8/, it has a discrete spectrum of eigenvalues $\gamma_i=\gamma_i(\mathrm{Ra})$, $i=0,1,2,\ldots$, and the eigenfunctions corresponding to them satisfy the normalization conditions

$$\int d\mathbf{r}\left\{\sum_{l=1}^{3}Y_{li}Y_{lj}+\mathrm{PrRa}Y_{4i}Y_{4j}\right\}=\delta_{ij},\quad i,j=0,1,2\ \ldots \tag{2.2}$$

The value of the Rayleigh number $\mathrm{Ra}=\mathrm{Ra}_c$, for which $\gamma_0(\mathrm{Ra}_c)=0$, determines the stability threshold of the system. When $\mathrm{Ra}<\mathrm{Ra}_c$, all the eigenvalues of problem (2.1) are positive /8/.

We shall use the stationary solutions of Eqs.(1.7) of the form

$$b_{\mu\nu}(\mathbf{r}_1,\mathbf{r}_2)=\sum_{i,j=0}^{\infty}b_{ij}Y_{\mu i}(\mathbf{r}_1)Y_{\nu j}(\mathbf{r}_2) \tag{2.3}$$

Let us substitute (2.3) into (1.7) taking account of (2.1). Then, by using the normalization condition (2.2), we obtain an algebraic equation for b_{ij}, the solution of which, allowing for (1.11), has the form

$$b_{ij} = \frac{Q \Pr \mathrm{Ra}}{\gamma_i(\mathrm{Ra}) + \gamma_j(\mathrm{Ra})} H_{ij}, \quad H_{ij} = \frac{5}{2} \int d\mathbf{r} T \left[Y_{3i} Y_{4j} + Y_{4i} Y_{3j}\right] \tag{2.4}$$

Let us now consider expressions (2.3) and (2.4) in the domain of values of Ra close to the minimum critical value Ra_c. Since $\gamma_0(\mathrm{Ra}_c) = 0$, the term with $i = j = 0$ makes the main contribution to the sum (2.3) in the case of a non-degenerate ground state level. By including in this term the first non-disappearing term of the expansion of $\gamma_0(\mathrm{Ra})$ in a Taylor's series about the value $\mathrm{Ra} = \mathrm{Ra}_c$, we obtain

$$b_{\mu\nu}(r_1, r_2) = \frac{\mathrm{Ra}_c}{\mathrm{Ra}_c - \mathrm{Ra}} \frac{Q \Pr \mathrm{Ra}}{2\alpha_0 \mathrm{Ra}_c} Y_{\mu 0}(\mathbf{r}_1) Y_{\nu 0}(\mathbf{r}_2) H_{00} \tag{2.5}$$

Here, $\alpha_0 = [- d\gamma_0/d\,\mathrm{Ra}]\,|_{\mathrm{Ra}=\mathrm{Ra}_c} > 0$. It follows from (2.5) that large scale fluctuations in a confined system close to the stability threshold are correlated over the whole of its volume. The synchronous spatial statistical links are anomalously reinforced in inverse proportion to the distance to the threshold of convective stability. This is identical with the result presented in /1, 4/. The nature of the anomalous reinforcement of the synchronous correlations turns out to be different in the case of systems with a continuous spectrum of eigenvalues of problem (2.1).

If the horizontal dimensions L of the system are substantially greater than the vertical dimensions, then, by analogy with /1/, we may put $j = (n, \mathbf{k})$ in (2.3) and (2.4)

$$Y_{\mu j}(\mathbf{r}) \equiv Y_{\mu n \mathbf{k}}(\mathbf{r}) = L^{-1} \exp(i\mathbf{k} \cdot \mathbf{r}^{\parallel}) Z_{\mu n}(z, \mathbf{k}), \quad n = 0, 1, 2, \ldots \tag{2.6}$$

Here, $\mathbf{k}$ and $\mathbf{r}''$ are vectors in the horizontal plane and z is the vertical coordinate.

Expressions for $b_{\mu\nu}$, analogous to (2.3) and (2.4) can be found by making use of the normalization condition for the functions (2.6)

$$\int d\mathbf{r} \left\{ \sum_{l=1}^{3} Y_{ln\mathbf{k}} Y_{ln'\mathbf{k}'} + \Pr \mathrm{Ra}\, Y_{4n\mathbf{k}} Y_{4n'\mathbf{k}'} \right\} = \delta_{nn'} \delta_{\mathbf{k}\mathbf{k}'}$$

As the horizontal dimensions of the system are increased the eigenvalue spectrum of problem (2.1) becomes more dense and, in the case of an infinitely large horizontal layer, passes into a spectrum which is continuous with respect to $\mathbf{k}$. Since, in this case, summation over $\mathbf{k}$ should be replaced by integration and it follows from (2.4) that

$$H_{ij} = \frac{4\pi^2}{L^2} H_{nm} \delta(\mathbf{k} + \mathbf{k}'), \quad H_{nm}(\mathbf{k}) = \frac{5}{2} \int dz T \left[Z_{4n} \bar{Z}_{3m} + Z_{3n} \bar{Z}_{4m}\right]$$

expressions (2.3) and (2.4) in this case are transformed into

$$b_{\mu\nu}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n, m=0}^{\infty} \int d\mathbf{k} \exp(i\mathbf{k} \cdot (\mathbf{r}_1^{\parallel} - \mathbf{r}_2^{\parallel})) b_{nm} Z_{\mu n} \bar{Z}_{\nu m} \tag{2.7}$$

$$b_{nm}(\mathbf{k}) = \frac{Q \Pr \mathrm{Ra}}{4\pi^2} \frac{H_{nm}(\mathbf{k})}{\gamma_n(\mathrm{Ra}, \mathbf{k}) + \bar{\gamma}_m(\mathrm{Ra}, \mathbf{k})}$$

Let us now consider representation (2.7) in the domain of values of the Rayleigh number close to the convection threshold. The solution of the equation $\gamma_n(\mathrm{Ra}_*, \mathbf{k}) = 0$ defines the neutral curves $\mathrm{Ra}_* = \mathrm{Ra}_{*n}(\mathbf{k})$ in the $(\mathrm{Ra}, \mathbf{k})$ plane which separate regions of stability and instability which, for any n, have a minimum. Let $\mathbf{k}_{nc}$ be the critical wave number for which

$$d\,\mathrm{Ra}_{*n}(k)/dk\,|_{k=k_{nc}} = 0, \quad d^2\,\mathrm{Ra}_{*n}(k)/dk^2\,|_{k=k_{nc}} > 0$$

Then, the minimum critical value of the Rayleigh number, which determines the threshold of convective stability, is

$$\mathrm{Ra}_c = \min_{n,\, k_{nc}} \mathrm{Ra}_{*n}(k) = \mathrm{Ra}_{*0}(k_c)$$

where $k_c \equiv k_{0c}$. In the region close to Ra_c

$$\gamma_0\,(\mathrm{Ra},\, \mathbf{k}) \approx \alpha_0\,(\mathbf{k})\,[\mathrm{Ra}_{*0}\,(\mathbf{k}) - \mathrm{Ra}]$$

and $\gamma_n \gg \gamma_0$ when $n = 1, 2, 3, \ldots$. Hence, in the sum in (2.7), the term with $n = m = 0$ as $\mathrm{Ra} \to \mathrm{Ra}_c$ is $[\mathrm{Ra}_c - \mathrm{Ra}]^{-1}$ times greater than the remaining terms in the series. By retaining terms in the series. By retaining just this dominant term, we obtain

$$b_{\mu\nu}(\mathbf{r}_1, \mathbf{r}_2) = \tag{2.8}$$

$$\frac{Q\,\mathrm{Pr}\,\mathrm{Ra}}{8\pi^2}\int d\mathbf{k}\,\frac{H_{00}(\mathbf{k})\exp(i\mathbf{k}\cdot(\mathbf{r}_1^{\parallel}-\mathbf{r}_2^{\parallel}))}{\alpha_0(\mathbf{k})\,[\mathrm{Ra}_{*0}(\mathbf{k})-\mathrm{Ra}]}\,Z_{\mu 0}(z,\mathbf{k})\,\bar{Z}_{\nu 0}(z_2,\mathbf{k})$$

We shall evaluate the integral in (2.8) allowing for the fact that, when $(\mathrm{Ra}_c-\mathrm{Ra})/\mathrm{Ra}_c\equiv\xi\ll 1$, a narrow region of k values close to k_c makes the main contribution to it. When this is so, by applying the asymptotic Laplace method for the case $k_c\,|\,\mathbf{r}_1''-\mathbf{r}_a''\,|\gg 1$, we obtain ($J_0$ is a zero-order Bessel function)

$$\begin{aligned} b_{\mu\nu}(\mathbf{r}_1,\mathbf{r}_2) &= B(k_c)\,R_c(\xi)\exp(-RR_c^{-1}(\xi))\,J_0(k_cR)\times\\ & Z_{\mu 0}(z_1,k_c)\,\bar{Z}_{\nu 0}(z_2,k_c),\quad R=|\,\mathbf{r}_1^{\parallel}-\mathbf{r}_2^{\parallel}\,|\\ B(k_c) &= Q\,\mathrm{Pr}\,\mathrm{Ra}\,k_c^3H_{00}(k_c)\,[4\,\mathrm{Ra}_c\,\alpha_0(k_c)]^{-1}\\ R_c(\xi) &= \frac{\Omega^{1/2}(k_c)}{\xi^{1/2}},\quad \Omega(k_c)=(2\,\mathrm{Ra}_c)^{-1}\frac{d^2\,\mathrm{Ra}_{*0}}{dk^2}\Big|_{k=k_c} \end{aligned} \tag{2.9}$$

The quantity $R_c(\xi)$ has the meaning of the correlation radius of the hydrodynamic fluctuations in the horizontal plane.

It follows from (2.9) that, close to the natural convection threshold in a horizontal layer of gas which is heated from below, the statistical relations propagate throughout the whole system in the vertical direction and up to a distance of the order of R_c in the horizontal direction. When $\xi\ll 1$, the intensity of the large-scale fluctuations and their correlation radius increase as $\xi^{-1/2}$.

This is identical with the result obtained by the Langevin method /1, 4/ and is associated with the fact that the values of the critical indices are determined by the spectral properties of the linearized operators $\Delta'_{\mu,\nu}$, which form the left-hand sides of the dynamic equations both in the Langevin method as well as in the method used here. At the same time, the coefficient $B(k_c)$ in (2.9) differs from the results obtained on the basis of the Langevin equations of hydrodynamics with random sources in the Landau and Lifshitz form since, as was shown in /5/, no account is taken of the contribution from inhomogeneous flows in the generation of the non-equilibrium fluctuations when such an approach is used.

Using the results which have been obtained, let us consider the time dynamics of large-scale fluctuations. In order to do this for the case of a confined cavity, we represent the solution of Eqs.(1.3) with initial conditions (1.4) in the form

$$\beta_{\mu\nu}(\tau,\mathbf{r}_1;0,\mathbf{r}_2)=\sum_{i,\,j=0}^{\infty} b_{ij}\exp(-\tau\gamma_i)\,Y_{\mu i}(\mathbf{r}_1)\,Y_{\nu j}(\mathbf{r}_2) \tag{2.10}$$

where the expansion coefficients b_{ij} are defined in (2.4). When $\xi\ll 1$, it follows from (2.10) and (2.4) that

$$\beta_{\mu\nu}(\tau,\mathbf{r}_1;0,\mathbf{r}_2)=\exp(-\tau\tau_c^{-1}(\xi))\,b_{\mu\nu}(\mathbf{r}_1,\mathbf{r}_2) \tag{2.11}$$

The coefficient $b_{\mu\nu}$ is defined in (2.5) and the quantity

$$\tau_c(\xi)=(\alpha_0\mathrm{Ra}_c\xi)^{-1} \tag{2.12}$$

is the attenuation time of the correlations. Formulae (2.11) and (2.12) describe the effect of the anomalous retardation of the large-scale hydrodynamic fluctuations close to the convection threshold.

Similar formulae can also be obtained for an infinite horizontal layer. In this case the coefficient $b_{\mu\nu}$ in these formulae is defined by expression (2.7) or (2.8), $\tau_c\to\tau_c'$, where $\tau_c'=\tau_c\,[1+(k-k_c)^2R_c^2]^{-1}$.

Let us now consider the dynamics of the small-scale fluctuations when thermal convection sets in and, in fact, we shall represent the two-time correlators $\alpha_{\mu\nu}$ in the form

$$\alpha_{\mu\nu}(\tau,\mathbf{r}_1;0,\mathbf{r}_2)=\sum_{i,\,j=0}^{\infty} a_{ij}\exp(-\tau\gamma_i)\,Y_{\mu i}(\mathbf{r}_1)\,Y_{\nu j}(\mathbf{r}_2) \tag{2.13}$$

By using (1.4) and (2.2) and the dimensionless form of expressions (1.6), we obtain

$$a_{ij}=Q\,\mathrm{Pr}\int d\mathbf{r}\left\{\frac{3}{2}TY_{li}Y_{lj}-D\left[Y_{3i}Y_{4j}+Y_{4i}Y_{3j}+\frac{3}{2}T^2\,\mathrm{Pr}\,\mathrm{Ra}\,Y_{4i}Y_{4j}\right]\right\},\quad D={}^2/_5\,\beta ghc_p^{-1}\ll 1 \tag{2.14}$$

In the critical region of Ra values the relation

$$\alpha_{\mu\nu}(\tau,\mathbf{r}_1;0,\mathbf{r}_2)=\exp\left(-\frac{\tau}{\tau_c}\right)\sum_{j=0}^{\infty}a_{0j}Y_{\mu 0}(\mathbf{r}_1)\,Y_{\nu j}(\mathbf{r}_2)+\sum_{i\geqslant 1,\,j=0}a_{ij}\exp(-\tau\gamma_i)\,Y_{\mu i}(\mathbf{r}_1)\,Y_{\nu j}(\mathbf{r}_2) \tag{2.15}$$

follows from (2.13). Formulae (2.12) and (2.15) describe the effect of the anomalous retardation of small-scale fluctuations close to the convection threshold (attenuating modes slowly develop when $i = 0$). However, unlike in (2.11), where, according to (2.5), $\beta_{\mu\nu} \sim Q\xi^{-1}$, the intensity of the small-scale fluctuations close to the convection threshold is, according to (2.15) and (2.14), $\alpha_{\mu\nu} \sim Q$ and hence $\alpha_{\mu\nu}\beta_{\mu\nu}^{-1} \sim \xi \ll 1$. It follows from this that it is possible, in general, to neglect the contribution from small-scale thermal noise in the process leading to the onset of convection.

We note that, in the theory in /2/, only the part of the non-equilibrium fluctuations, $\alpha_{\mu\nu}$, was calculated while the part $\beta_{\mu\nu}$ was completely neglected. In this case the conclusion drawn in /2/ concerning the insensitivity of the intensity of the hydrodynamic fluctuations towards the development of natural convection is erroneous since, in fact, the intensity of the large-scale fluctuations increases when $\xi \ll 1$.

The dependence of the intensity of the long-range correlations on the parameters of the system which has been found is identical with the result in /3/ which was based on kinetic theory. This is dictated by the fact that the equations for the spatial correlations which are used in this paper are the hydrodynamic asymptotic solutions of the kinetic equations given in /3/.

3. Critical opalescence during the onset of thermal convection. We shall show that the anomalous reinforcement of the intensity of the large-scale fluctuations in the hydrodynamic fields when $\xi \ll 1$ is the cause of the sharp increase in the intensity of the scattered light. For the analysis of the radiation which is scattered by a planar layer of a gas which is heated from below we shall calculate the two-time, two-point correlation function of the fluctuations in the permittivity $\delta\varepsilon$ of the scattering medium. We shall confine the vibrational frequenct range to conditions under which the Oberbeck-Boussinesq approximations are applicable /8/. Since $\delta\rho = -\rho_0\beta\delta T$ in this approximation, it follows from the equation $\delta\varepsilon = (\delta\varepsilon/\delta\rho)_T\delta\rho + (\delta\varepsilon/\delta T)_\rho\delta T$ that $\delta\varepsilon \sim \delta T$ and the final expression for the correlation function can be immediately written down:

$$\begin{aligned}
&\langle\delta\varepsilon\,(t+\tau,\mathbf{r}_1)\delta\varepsilon\,(t,\mathbf{r}_2)\rangle = c\,(z_1)c\,(z_2)\,\langle\delta T\,(t+\tau,\mathbf{r}_1)\delta T\,(t,\mathbf{r}_2)\rangle = \\
&\quad c\,(z_1)c\,(z_2)(ah)^2\,[\alpha_{44}\,(t+\tau,\mathbf{r}_1;t,\mathbf{r}_2) + \beta_{44}\,(t+\tau,\mathbf{r}_1;t,\mathbf{r}_2)] \\
&c\,(z) = |(\delta\varepsilon/\delta T)_\rho - \rho_0\beta\,(\delta\varepsilon/\delta\rho)_T\,|
\end{aligned} \tag{3.1}$$

where, as before, the correlators α_{44}, β_{44} are dimensionless. Hence, in the Oberbeck-Boussinesq approximation we are dealing with thermal waves.

The sharp increase in the intensity of the fluctuations in the immediate vicinity of the convection threshold generates chaotic convective motion of macroscopic volumes of the medium. Nevertheless, in the neighbourhood $\xi = 0$, it is possible to pick out a domain of values of ξ where the intensity of the fluctuations is still not very large and the Born approximation for the modelling of the electromagnetic field within an optical inhomogeneity is valid. In this region, the spectrum of the single-pass scattering of light by a volume V of an inhomogeneous medium in a direction $\eta = \mathbf{r}/r$ when $|\mathbf{r}| \gg V^{1/3}$ is defined by the expression /9/

$$\begin{aligned}
&I(\omega,\eta) = \operatorname{Re}\Big\{M\int\limits_{-\infty}^{\infty}d\tau\int\limits_V d\mathbf{r}_1\int d\mathbf{r}_2\exp\{i[\tau(\omega-\omega_0)+\varkappa\cdot(\mathbf{r}_1-\mathbf{r}_2)]\}\times \\
&\quad\langle\delta\varepsilon\,(t+\tau,\mathbf{r}_1)\,\delta\varepsilon\,(t,\mathbf{r}_2)\rangle\Big\} \approx \operatorname{Re}\Big\{M'\int\limits_{-\infty}^{\infty}d\tau\int\limits_V d\mathbf{r}_1\int\limits_V d\mathbf{r}_2\times \\
&\quad c\,(z_1)\,c\,(z_2)\,\beta_{44}\,(t+\tau,\mathbf{r}_1;t,\mathbf{r}_2)\exp\{i[\tau(\omega-\omega_0)+\varkappa\cdot(\mathbf{r}_1-\mathbf{r}_2)]\}\Big\} \\
&M = I_0\,|\varkappa_0|^4\left[\frac{\sin\zeta}{(2\pi)^{3/2}\,r}\right]^2,\quad M' = \frac{h^2}{\nu}M,\quad I_0 = \frac{c}{8\pi}E_0^2 \\
&\varkappa = \varkappa_0 - \varkappa_p,\quad \varkappa_p = \frac{\omega}{c}\eta
\end{aligned} \tag{3.2}$$

Here, $\operatorname{Re}\{\cdot\}$ is the real part of the expression $\{\cdot\}$, ζ is the angle between the direction of polarization of the incident wave and the direction of observation and ω_0, $\varkappa_0$, I_0 are the frequency, the wave vector and the intensity of the exciting radiation. The integrand in the second equality of (3.2) is written in dimensionless variables (in doing this, the previous notation is retained for the dimensionless parameters).

Let us now consider the scattering of a beam of exciting radiation with a transverse cross-section in the form of a circle with an area S which is incident at an angle σ on the plane of the layer. We shall rotate the coordinate system introduced in Sect.1 in such a way that the axis of the beam lies in the x, z-plane. By substituting Eqs.(2.10) and (2.8) into (3.2), we have in the case of values of the angle σ that are not too small

$$I(\omega,\eta) = \operatorname{Re}\Big\{M'\frac{SQ\,\mathrm{Pr}\,\mathrm{Ra}}{8\pi^2\sin\sigma}\int\limits_{-\infty}^{\infty}d\tau\int\limits_{S_0}d\mathbf{R}\int d\mathbf{k}\,\times \tag{3.3}$$

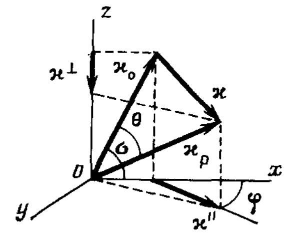

Fig.1

$$\exp\{i[\tau(\omega-\omega_0)+\mathbf{R}\cdot(\varkappa^{\parallel}-\mathbf{k})]\}\exp\{-\tau\gamma_0(\xi,\mathbf{k})\}\times$$
$$H_{00}(\mathbf{k})\,\Xi_{00}(\varkappa^{\perp},\mathbf{k})\,[\alpha_0(\mathbf{k})(\mathrm{Ra}_{*0}(\mathbf{k})-\mathrm{Ra})]^{-1}$$
$$\Xi_{00}(\varkappa^{\perp},\mathbf{k})=\int_0^1 dz_1\,dz_2\,c(z_1)\,c(z_2)\exp\{-i\varkappa^{\perp}(z_1-z_2)\}\,Z_{40}(z_1)\,\bar{Z}_{40}(z_2)$$

Here S_0 is the area of the horizontal cross-section of the scattering volume, $\varkappa^{\parallel}=\varkappa^{\parallel}(\theta,\sigma,\varphi)$ is the projection of the vector $\varkappa$ onto the horizontal plane and $\varkappa^{\perp}=\varkappa^{\perp}(\theta,\sigma,\varphi)$ is the projection of the vector $\varkappa$ onto the Oz-axis (Fig.1).

Let us first evaluate the integral with respect to $\mathbf{R}$. In doing this, we take account of the fact that the magnitude of the optical inhomogeneity R_c attains macroscopic dimensions when $\xi\ll 1$. We shall consider the case when a broad beam of radiation is incident on a planar layer of liquid which is heated from below, that is, when the condition $(S_0)^{1/2}\gg R_c$ is satisfied. In this case the integral with respect to $\mathbf{R}$, when account is taken of the asymptotic behaviour of (2.9), can be extended onto the whole of the horizontal plane

$$\int_{S_0} d\mathbf{R}\exp\{i\mathbf{R}\cdot(\varkappa^{\parallel}-\mathbf{k})\}\approx\int_{\infty} d\mathbf{R}\exp\{i\mathbf{R}\cdot(\varkappa^{\parallel}-\mathbf{k})\}=(2\pi)^2\,\delta(\varkappa^{\parallel}-\mathbf{k})$$

When this is done, the following equation follows from (3.3):

$$I(\omega,\eta)=I(\varkappa^{\perp},|\varkappa^{\parallel}|)\frac{\gamma_0(\xi,|\varkappa^{\parallel}|)}{(\omega-\omega_0)^2+\gamma_0^2(\xi,|\varkappa^{\parallel}|)}\,\frac{1}{\mathrm{Ra}_{*0}(|\varkappa^{\parallel}|)-\mathrm{Ra}} \tag{3.4}$$
$$I(\varkappa^{\perp},|\varkappa^{\parallel}|)=M'SQ\,\mathrm{Pr}\,\mathrm{Ra}\,H_{00}(|\varkappa^{\parallel}|)\,\Xi_{00}(\varkappa^{\perp},|\varkappa^{\parallel}|)$$

Close to the convection threshold the spectral distribution of the intensity of the scattered radiation has a sharp peak in the region of values of the wave number $|\varkappa^{\parallel}|$ close to k_c. In fact, since, when

$$||\varkappa^{\parallel}|-k_c|/k_c\ll 1 \tag{3.5}$$

we have

$$\mathrm{Ra}_{*0}(|\varkappa^{\parallel}|)\approx\mathrm{Ra}+(\mathrm{Ra}_c-\mathrm{Ra})\,q(|\varkappa^{\parallel}|)$$
$$\gamma_0(|\varkappa^{\parallel}|)\approx\tau_c^{-1}q(|\varkappa^{\parallel}|),\quad q(|\varkappa^{\parallel}|)=1+(|\varkappa^{\parallel}|-k_c)^2R_c^2$$

whereupon it follows from (3.4) that

$$I(\omega,\eta)=C\,\frac{\tau_c^{-1}q}{(\omega-\omega_0)^2+\tau_c^{-2}q^2}\,\frac{R_c^{\,2}}{1+(|\varkappa^{\parallel}|-k_c)^2R_c^{\,2}} \tag{3.6}$$
$$C=\frac{I(\varkappa^{\perp},k_c)}{\Omega(k_c)\,\mathrm{Ra}_c}$$

Hence, the anomalously increasing large-scale hydrodynamic fluctuations give rise to the Lorentzian lineshape for the scattered radiation which becomes narrower and narrower without constraint when $\xi\to 0$. When $|\varkappa^{\parallel}|=k_c$, its intensity increases in inverse proportion to the distance to the stability threshold. The specific angular dependence of the non-equilibrium critical opalescence in expression (3.6) is associated with the occurrence, when $\xi\ll 1$, of optical anisotropy of the planar layer and the existence of a finite lower limit for the wave numbers k_c of the anomalous fluctuations. This dependence is the unique difference between the characteristic terms in (3.6) and the characteristic terms of the corresponding expression in /9/ for the intensity of the scattered light close to the critical points of equilibrium systems.

We shall investigate the angular dependence of the anomalous scattering. The explicit expression for $|\varkappa^{\parallel}|$ has the form

$$|\varkappa^{\parallel}(\theta,\sigma,\varphi)|=2|\varkappa_0|\sin(^1/_2\theta)\,[1-\cos^2\sigma\sin^2\varphi]^{-1}\times \tag{3.7}$$
$$(\sin\sigma\,[\cos^2(^1/_2\theta)-\cos^2\sigma\sin^2\varphi]^{1/2}-\sin(^1/_2\theta)\cos\sigma\cos\varphi)$$

(the angles θ,σ and φ are shown in Fig.1). In particular, it follows from (3.7) that, when $\sigma\to\pi/2$, we have $|\varkappa^{\parallel}|\to|\varkappa_0|\sin\theta$ and it is independent of the angle φ.

Let us now consider the condition $|\varkappa^{\parallel}(\theta_c,\sigma,f)|=k_c$ which determines the directions of the maxima in the anomalous scattering map. Taking account of (3.7), we find

$$2\sin^2(^1/_2\theta_c)=-\theta_0\cos\sigma\cos\varphi+\sin^2\sigma\left[1-\left(1-\frac{\theta_0^2+2\theta_0\cos\sigma\cos\varphi}{\sin^2\sigma}\right)\right]^{1/2},\quad \theta_0=\frac{k_c}{|\varkappa_0|}=\frac{\lambda_0}{\lambda_c} \tag{3.8}$$

Here, $\lambda_0 = 2\pi|\varkappa_0|^{-1}$ is the wavelength of the incident radiation, $\lambda_c = 2\pi k_c^{-1}$ is the wavelength of the critical fluctuations (for the case when both boundaries of the layer are solid and absolutely thermally conducting, $\lambda_c \approx 2h$ /8/). In the case of optical radiation with $\lambda_0 \sim 10^{-7}$ m incident of layer of liquid with a thickness, $h \sim 10^{-3}$ m confined by solid walls, the quantity $\theta_0 \sim 10^{-4} \ll 1$. At the same time, we find $\theta_c \approx \theta_0 = \lambda_0/\lambda_c$ from (3.8), that is, in the given case the angle of maximum scattering is very small. As λ_0 is increased up to $\lambda = \lambda_c f(\sigma, \varphi)$, where

$$f(\sigma, \varphi) = -\cos\sigma\cos\varphi + [\cos^2\sigma\cos^2\varphi + \sin^2\sigma]^{1/2}$$

the magnitude of θ_c increases up to macroscopic values of the angles. In the wavelength region $\lambda_0 \gg \lambda$, inequality (3.5) is not satisfied and therefore, according to (3.4), such waves do not experience anomalous scattering close to the convection threshold.

Hence, when light is scattered by a planar layer of liquid which is close to the natural convection threshold, a phenomenon is to be expected which is analogous to critical opalescence during equilibrium phase transitions of the second kind. At the same time an angular effect should be detected which is associated with the existence of a finite upper limit of the wavelengths of the critical fluctuations λ_c in an inhomogeneous system. The value of the "angle of non-equilibrium critical opalescence" θ_c is determined by the wavelength of the incident radiation. In the region of radiation wavelengths which are comparable with λ_c, this angle reaches macroscopic values.

REFERENCES

1. ZAITSEV B.M. and SHLIOMIS M.I., Hydrodynamic fluctuations close to the convection threshold, Zh. Eksp. Teor. Fiz., 59, 5, 1970.
2. LESNIKOV V.P. and FISHER I.Z., Thermal fluctuations and molecular scattering of light in a liquid close to a state of convective instability, Zh. Eksp. Teor. Fiz., 67, 4, 1974.
3. KIRKPATRICK T.R. and COHEN E.G.D., Kinetic theory of fluctuations near a convective instability, J. Statist. Phys., 33, 3, 1983.
4. BELYI V.V., On the hydrodynamic fluctuations in an inhomogeneously heated gas, Teoret. i mat. Fiz., 58, 3, 1984.
5. GRECHANNYI O.A. and TOKARCHUK V.V., On the kinetic theory of thermal hydrodynamic fluctuations in an inhomogeneous gas, Prikl. Mat. i Mekhan., 45, 1, 1981.
6. GRECHANNYI O.A. and TOKARCHUK V.V., Dynamics of the space-time correlations of the hydrodynamic fields in non-stationary gas flows, Prikl. Mat. i Mekhan., 49, 4, 1985.
7. GRECHANNYI O.A. and TOKARCHUK V.V., Thermal hydrodynamic fluctuations near a state of convective Rayleigh-Benard instability, in: Convective Heat Transfer (Konvektivnyi teploperenos), Naukova Dumka, Kiev, 1982.
8. GERSHUNI G.Z. and ZHUKOVITSKII B.M., Convective Stability of an Incompressible Liquid (Konvektivnaya ustoichivost' neszhimayemoi zhidkosti), Nauka, Moscow, 1972.
9. CROSIGNANI B., DI PORTO B. and BERTOLOTII M., Statistical Properties of Scattered Light, Nauka, Moscow, 1980.

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